PROPAGATION OF AN INITIAL PULSE IN RELATIVISTIC MAGNETO-HYDRODYNAMICS WITH FINITE CONDUCTIVITY

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The problem of the propagation of a small initial pulse in nonrelativistic magnetohydrodynamics with finite conductivity was solved completely by G. S. Golitsyn [1]. The solution proved to be relatively easy since the displacement current may be ignored in the Maxwell equations (the displacement current, as will be seen below, is of the order $1/c^2$ in comparison with the remaining terms). The fact that the displacement current is small makes it possible to express the electric field in terms of the magnetic field and hence reduce the number of necessary equations to a minimum. Here, the Alfvén and magnetosonic waves are described by two independent systems of equations which indicate that they propagate without interaction even at finite conductivity.

In relativistic magnetohydrodynamics, the displacement current is of the same order as the remaining terms and therefore cannot be omitted. To solve the problem it is necessary to consider a complete system of relativistic magnetohydrodynamic equations and Maxwell equations, in which both the electric and magnetic fields appear simultaneously. In the general relativistic case, the Alfvén and magnetosonic waves are not separate.

Let us consider the Maxwell equations with relativistic current on the right side [2-4]:

$$\operatorname{rot}\mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\gamma\lambda\left(\mathbf{E} + \frac{1}{c}[\mathbf{vH}]\right), \quad \gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2},$$
$$\operatorname{div}\mathbf{E} = \frac{4\pi}{c^2}\gamma\lambda(\mathbf{vE}), \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \qquad \operatorname{div}\mathbf{H} = 0. \tag{1}$$

Here it is assumed that there are no free charges, and that $\varepsilon = \mu = 1$.

The Maxwell equations in the form (1) are not suitable for linearizing the magnetohydrodynamic equations. Therefore, we rewrite equations (1) in a form similar to that of the equations of skin-effect theory in electrodynamics. Applying the operation to both parts of the first and third equations in system (1), we can finally write the necessary system of equations in the form

$$nu_k\frac{\partial (wu_i)}{\partial x_k} + \frac{\partial p}{\partial x_i} + \frac{\partial T_{i\beta}}{\partial x_\beta} + \frac{\partial T_{i4}}{\partial x_4} = 0,$$
$$T_{\alpha\beta} = \frac{1}{4\pi}\left\{-H_\alpha H_\beta - E_\alpha E_\beta + \frac{1}{2}\delta_{\alpha\beta}(E^2 + H^2)\right\},$$
$$T_{\alpha 4} = \frac{i}{4\pi}[\mathbf{EH}]_\alpha, \quad T_{44} = -\frac{1}{8\pi}(E^2 + H^2) \quad (\alpha, \beta = 1, 2, 3).$$
$$\frac{\partial (nu_k)}{\partial x_k} = 0,$$
$$\operatorname{div}\mathbf{H} = 0, \qquad \operatorname{rot}\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}, \tag{2}$$
$$\nu_m\left(\Delta\mathbf{H} - \frac{1}{c^2}\frac{\partial^2 \mathbf{H}}{\partial t^2}\right) - \gamma\frac{\partial \mathbf{H}}{\partial t} + \operatorname{rot}(\gamma\mathbf{v}\times\mathbf{H}) - c\,[\mathbf{E}\operatorname{grad}\gamma] = 0$$
$$\left(\nu_m = \frac{c^2}{4\pi\lambda}\right),$$
$$\nu_m\left(\Delta\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2}\right) - \operatorname{grad}\gamma(\mathbf{v}\cdot\mathbf{E}) - \frac{\partial}{\partial t}\gamma\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{H}\right) = 0.$$

Here T_{ik} is the electromagnetic field tensor, ν_m is the magnetic viscosity of the medium, w is the thermal function for one particle, and n is the number of particles per unit volume.

System (2) together with the equation of state forms a closed system. Note that not all the equations in system (2) are independent, since the components rot **H** and rot **E** are linked by the relations div rot **H** = 0 and div rot **E** = 0.

For the linearized problem in the case of a still medium ($v_\alpha = 0$), system (2) takes the form

$$\frac{nw}{c^2}\frac{\partial \delta v_\alpha}{\partial t} + w\frac{c_0^2}{c^2}\frac{\partial \delta n}{\partial x_\alpha} + \frac{1}{4\pi}\left\{-H_\alpha\frac{\partial h_\beta}{\partial x_\beta} - H_\beta\frac{\partial h_\alpha}{\partial x_\beta} - E_\alpha\frac{\partial e_\beta}{\partial x_\beta} - E_\beta\frac{\partial e_\alpha}{\partial x_\beta} + \right.$$
$$\left. + E_\beta\frac{\partial e_\beta}{\partial x_\alpha} + H_\beta\frac{\partial h_\beta}{\partial x_\alpha} + \frac{1}{c}\left(\mathbf{E}\times\frac{\partial \mathbf{h}}{\partial t}\right)_\alpha + \frac{1}{c}\left(\frac{\partial \mathbf{e}}{\partial t}\times\mathbf{H}\right)_\alpha\right\} = 0,$$
$$n\frac{\partial \delta v_\alpha}{\partial x_\alpha} + \frac{\partial \delta n}{\partial t} = 0, \quad \frac{\partial h_\beta}{\partial x_\beta} = 0, \quad -\frac{1}{c}\frac{\partial \mathbf{h}}{\partial t} = \operatorname{rot}\mathbf{e}, \tag{3}$$
$$\nu_m\left(\Delta\mathbf{h} - \frac{1}{c^2}\frac{\partial^2 \mathbf{h}}{\partial t^2}\right) - \frac{\partial \mathbf{h}}{\partial t} + \operatorname{rot}(\delta\mathbf{v}\times\mathbf{H}) = 0.$$
$$\nu_m\left(\Delta\mathbf{e} - \frac{1}{c^2}\frac{\partial^2 \mathbf{e}}{\partial t^2}\right) - \operatorname{grad}(\delta\mathbf{v}\mathbf{E}) - \frac{\partial}{\partial t}\left(\mathbf{e} + \frac{1}{c}\delta\mathbf{v}\times\mathbf{H}\right) = 0.$$

Here

$$\frac{\partial \delta p}{\partial x_\alpha} = w\frac{c_0^2}{c^2}\frac{\partial \delta n}{\partial x_\alpha}, \qquad \frac{c_0^2}{c^2} = \frac{\gamma_0 p}{\varepsilon}\frac{1}{1 + p/\varepsilon}$$

(since joule losses can be ignored for small perturbations), ε is the internal energy density per unit volume, and γ_0 is the adiabatic exponent.

We shall represent small perturbations of velocity, density, and electric and magnetic field components in the form of a super-position of traveling waves

$$\delta n(x, t) = \int_{-\infty}^{\infty} \delta n^\circ(k)\, e^{i(kx - \omega(k)t)}\, dk, \ldots \tag{4}$$

where $\delta n^\circ(k)$, $\delta v^\circ(k)$, etc. are given at the initial time. Then from from the third equation of system (3) we have

$$h_\beta^\circ k_\beta = 0. \tag{5}$$

Therefore, we can choose the coordinate system in which the magnetic field perturbation components and the wave vector have the form $h_\beta^\circ(0, h_y^\circ, h_z^\circ)$, $k_\beta = (k, 0, 0)$.

In the chosen system, proceeding from the fourth of equations (3), we have

$$e_y^\circ = \frac{\omega}{kc}h_z^\circ, \qquad e_z^\circ = -\frac{\omega}{kc}h_y^\circ. \tag{6}$$

The e_x° component remains arbitrary. Substituting (4) into (3) and allowing for conditions (5) and (6), we obtain the homogeneous system

$$\{(4\pi W)(\Omega^2 - a^2)[\Omega + \beta'(1 - \Omega^2)] - \Omega(1 - \Omega^2)(H_y^2 + H_z^2) +$$
$$+ \Omega E_x^2\}\delta v_x^\circ + \Omega\{(1 - \Omega^2)H_xH_y + E_x(E_y - \Omega H_z)\delta v_y^\circ +$$
$$+ \Omega\{(1 - \Omega^2)H_xH_z + E_x(E_z + \Omega H_y)\}\delta v_z^\circ = 0, \tag{7}$$
$$\{(1 - \Omega^2)H_xH_y + E_x(E_y - \Omega H_z)\}\delta v_x^\circ + \{(4\pi W)\Omega[\Omega + \beta'(1 - \Omega^2)] -$$
$$- (1 - \Omega^2)H_x^2 + (E_y - \Omega H_z)^2\}\delta v_y^\circ +$$
$$+ (E_y - \Omega H_z)(E_z + \Omega H_y)\delta v_z^\circ = 0,$$
$$\{(1 - \Omega^2)H_xH_z + E_x(E_z + \Omega H_y)\}\delta v_x^\circ +$$
$$+ (E_y - \Omega H_z)(E_z + \Omega H_y)\delta v_y^\circ +$$
$$+ \{(4\pi W)\Omega[\Omega + \beta'(1 - \Omega^2)] - (1 - \Omega^2)H_x^2 + (E_z + \Omega H_y)^2\}\delta v_z^\circ = 0$$

for the Fourier amplitudes δv_x°, δv_y° and δv_z°.

The amplitudes e_x°, h_y°, h_z° are related to δv_x°, δv_y°, δv_z° by the formulas

$$e_x^\circ = \frac{1}{c[\Omega + \beta'(1-\Omega^2)]}[E_x \delta v_x^\circ + (E_y - \Omega H_z)\delta v_y^\circ + (E_z + \Omega H_y)\delta v_z^\circ],$$
$$h_y^\circ = \frac{H_y \delta v_x^\circ - H_x \delta v_y^\circ}{c[\Omega + \beta'(1-\Omega^2)]}, \qquad h_z^\circ = \frac{H_z \delta v_x^\circ - H_x \delta v_z^\circ}{c[\Omega + \beta'(1-\Omega^2)]}. \quad (8)$$

The following notation has been introduced (7) and (8):

$$a^2 = \frac{c_0^2}{c^2}, \quad W = nw, \quad \Omega = \frac{\omega}{kc}, \quad \beta' = i\beta = i\frac{v_m k}{c}.$$

Here Ω is the dimensionless frequency and β the dimensionless magnetic viscosity. The requirement that there exist a nontrivial solution to system (7) leads to a tenth-degree dispersion equation in Ω:

$$a_1\Omega^{10} + a_2\Omega^9 + \ldots + a_{10}\Omega + a_{11} = 0. \quad (9)$$

Here, the coefficients $a_1, a_2, \ldots, a_{11}$ are functions of the electric and magnetic field strengths and of a^2 and β^1. For infinite conductivity, as might be expected, the dispersion equation (9) reduces to the two expressions*

$$\Omega^2\left(1 + \frac{H^2}{4\pi W}\right) - \frac{H_x^2}{4\pi W} = 0,$$
$$\Omega^4\left(1 + \frac{H^2}{4\pi W}\right) - \Omega^2\left[\frac{H^2}{4\pi W} + a^2\left(1 + \frac{H_x^2}{4\pi W}\right)\right] + a^2\frac{H_x^2}{4\pi W} = 0. \quad (10)$$

These equations describe Alfvén and magnetosonic waves. For $\beta \ll 1$, these waves are approximately separate and the phase velocities have the form:

for Alfvén waves

$$\Omega_{1,2} = \pm\frac{H_x}{\sqrt{4\pi W}\sqrt{1 + H^2/4\pi W}} + \frac{[EH]_x}{4\pi W} - i\beta\left\{\frac{1}{2}\left(1 - \frac{H^2}{4\pi W}\right) + \frac{(1 + a^2H^2/4\pi W)H_x^2/4\pi W}{2(a^2-1)(1 + H^2/4\pi W)}\right\};$$

for magnetosonic waves

$$\Omega = \pm\frac{1}{2}\left\{\left(\frac{H^2/4\pi W + a^2(1 + H_x^2/4\pi W)}{1 + H^2/4\pi W} + \frac{aH_x/\sqrt{\pi W}}{\sqrt{1 + H^2/4\pi W}}\right)^{1/2} \pm \left(\frac{H^2/4\pi W + a^2(1 + H^2/4\pi W)}{1 + H^2/4\pi W} - \frac{aH_x/\sqrt{\pi W}}{\sqrt{1 + H^2/4\pi W}}\right)^{1/2}\right. - i\beta B_{1,2}(E, H).$$

Consider the case of low conductivity ($\beta \gg 1$). The solution of dispersion equation (9) is sought in the form

$$\Omega = \Omega_0 + \frac{1}{\beta'}\Omega_1,$$

where Ω_0 and Ω_1 satisfy the equations

$$\Omega_0\{\Omega_0^6 - (a^2\theta + 2)\Omega_0^4 + (2a^2\theta + 1)\Omega_0^2 - a^2\theta\} = 0, \quad (13)$$
$$\theta = 1 - \frac{E_x^2}{E^2}, \quad \Omega_1 = \frac{b_0\Omega_0^{10} + b_1\Omega_0^8 + b_2\Omega_0^6 + b_3\Omega_0^4 + b_4\Omega_0^2 + b_5}{c_0\Omega_0^6 + c_1\Omega_0^4 + c_2\Omega_0^2 + c_3}. \quad (14)$$

*Equations (10) are a particular case of the more general dispersion equations for a moving medium [3].

Here $b_0, \ldots, b_5$, $c_0, \ldots, c_5$ are functions of E, H, a^2. For one of the roots of equation (13) ($\Omega_0 = 0$) we have the wave

$$\Omega^{(1)} = -\frac{i}{\beta}\frac{H_x^2}{4\pi W},$$

which rapidly attenuates with time. The remaining six roots are found from the solution of the equation

$$\chi'^3 + 3p\chi' + 2q = 0. \quad (15)$$

Here

$$\Omega_0 = \pm\sqrt{\chi}, \quad \chi = \chi' + {}^1/_3(a^2\theta + 2),$$
$$p = -{}^1/_9(1 - a^2\theta)^2, \quad q = -{}^1/_{27}\{4a^6\theta^3 + 15a^4\theta^2 + {}^{105}/_2 a^2\theta + 23\}. \quad (15a)$$

Waves described by this equation will also have the form

$$\Omega = P - \frac{i}{\beta}Q, \quad (15b)$$

where P and Q depend upon the value of the roots of equation (15) and the magnitudes of the electric and magnetic field strengths. From these waves we must select only those that attenuate with time. For the limit case under consideration $\beta \gg 1$, separation of the waves into Alfvén and magnetosonic waves makes no sense.

It is of interest to consider the trivial solution to system (7) when initially $\delta v_x^\circ = \delta v_y^\circ = \delta v_z^\circ = 0$. For the electric and magnetic field perturbations to be finite, it is necessary to set

$$\Omega + \beta'(1 - \Omega^2) = 0. \quad (16)$$

It is clear that (16) describes the ordinary skin absorption of electromagnetic waves with frequency

$$\Omega = -i/\beta \quad \text{for} \quad \beta \ll 1, \qquad \Omega = \pm 1 - {}^1/_2 i/\beta \quad \text{for} \quad \beta \gg 1.$$

If we set $\beta \to \infty$ in equation (16), we have $\Omega \to \pm 1$, which corresponds to electromagnetic wave propagation in a dielectric with $\varepsilon = \mu = 1$.

In the general case ε and μ are neither equal to unity nor constant with time, and the wave propagation velocity must be given by

$$\omega/k = c/\sqrt{\varepsilon\mu}.$$

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